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# Example of zero viscosity limit for two dimensional nonstationary Navier-Stokes flows with boundary

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## 1. INTRODUCTION

Our purpose in this report is to give an example for a flow  $u^\nu$  of the nonstationary, incompressible Navier-Stokes equations in  $\Omega$  which is convergent to an Euler flow  $\bar{u}$  as the viscosity  $\nu$  tends to zero, where  $\Omega$  is a bounded domain with smooth boundary.

Let  $(u^\nu(t), p^\nu(t))$  be the unique global classical solution of the Navier-Stokes equations for the viscosity  $\nu$  (we omit a variable  $x \in \Omega$  for simplicity):

$$\begin{aligned} (NS) \quad & u_t^\nu - \nu \Delta u^\nu + (u^\nu, \nabla) u^\nu + \nabla p^\nu = f^\nu, \\ & \operatorname{div} u^\nu = 0 \quad \text{in } \Omega \times (0, T), \\ & u^\nu|_{\partial\Omega} = 0, \quad u^\nu|_{t=0} = u_0^\nu, \end{aligned}$$

where  $f^\nu(t)$  and  $u_0^\nu$  are outer forces and initial data which satisfy the compatibility conditions  $u_0^\nu|_{\partial\Omega} = 0$  and  $\operatorname{div} u_0^\nu = 0$  (for existence, see [3]). If  $\nu \rightarrow 0$  in (NS) formally, we have the Euler equations which has the unique global classical solution  $(\bar{u}(t), \bar{p}(t))$  (see, [1]):

$$\begin{aligned} (EE) \quad & \bar{u}_t + (\bar{u}, \nabla) \bar{u} + \nabla \bar{p} = \bar{f}, \\ & \operatorname{div} \bar{u} = 0 \quad \text{in } \Omega \times (0, T), \\ & \bar{u} \cdot n|_{\partial\Omega} = 0, \quad \bar{u}|_{t=0} = \bar{u}_0, \end{aligned}$$

where  $\bar{f}(t)$ ,  $\bar{u}_0$  and  $n$  are outer forces, initial data and the unit outer normal to  $\partial\Omega$  respectively with  $\bar{u}_0$  satisfying the compatibility conditions  $\bar{u}_0 \cdot n|_{\partial\Omega} = 0$  and  $\operatorname{div} \bar{u}_0 = 0$ .

To prove the convergent of our flow in the example we need

**THEOREM 1.** Assume

- (1)  $u_0^\nu \rightarrow \bar{u}_0$  as  $\nu \rightarrow 0$  in  $L^2(\Omega)$ ,
- (2)  $f^\nu \rightarrow \bar{f}$  as  $\nu \rightarrow 0$  in  $L^1(0, T; L^2(\Omega))$ .

Then the following three conditions are equivalent for  $t \in [0, T]$ .

- (a)  $\|u^\nu(t) - \bar{u}(t)\|_{L^2(\Omega)} \rightarrow 0$  as  $\nu \rightarrow 0$  uniformly (pointwisely) in  $t$ ,
- (b)  $\overline{\lim}_{\nu \rightarrow 0} \nu \int_0^t \int_{\partial\Omega} \bar{u}(\tau) \cdot n \times \text{rot } u^\nu(\tau) dS d\tau = 0$  uniformly (pointwisely) in  $t$ ,
- (c)  $\lim_{\nu \rightarrow 0} \nu \int_0^t \int_{\partial\Omega} \bar{u}(\tau) \cdot n \times \text{rot } u^\nu(\tau) dS d\tau = 0$  uniformly (pointwisely) in  $t$ ,

where  $dS$  denotes surface area of  $\partial\Omega$ ,  $\text{rot } u = \partial u_2 / \partial x_1 - \partial u_1 / \partial x_2$  for vector fields  $u(x) = (u_1(x), u_2(x))$  in  $x = (x_1, x_2)$  and  $a \times b = (a_2 b, -a_1 b)$  for a vector  $a = (a_1, a_2)$  and a scholar  $b$ .

Remark. (1) Shiota also obtained Theorem 1 in somewhat different statements independently of ours, which is not published.

(2) Kato[2] obtained other equivalent conditions to (a) for the flows in a bounded domain of  $\mathbb{R}^n$ . One of them is

$$\nu \int_0^T \|\text{grad } u^\nu(\tau)\|_{L^2(\Gamma_{c\nu})}^2 d\tau \rightarrow 0 \text{ as } \nu \rightarrow 0,$$

where  $\Gamma_{c\nu}$  is the boundary strip of width  $c\nu$  with  $c > 0$  fixed.

## 2. EXAMPLE

In this section  $\Omega$  is the open unit disk  $\{x = (x_1, x_2) \in \mathbb{R}^2; |x| = (x_1^2 + x_2^2)^{1/2} < 1\}$ . For simplicity we denote  $r = |x|$  and  ${}^t(\cos \theta, \sin \theta) = x/|x|$ , where  ${}^t(\cdot, \cdot)$  is a transported vector of  $(\cdot, \cdot)$ . We note that the unit outer normal to  $\partial\Omega$  is  $x/|x|$ . Furthermore we assume  $f^\nu = \bar{f} = 0$ .

We employ the stationary solution  $\bar{u}$ , defined by a rotating eddy, to the Euler equations (see, [4]):

$$(2.1) \quad \bar{u}(x) (= \bar{u}_0(x)) = \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix} \frac{1}{r} \int_0^r \rho \bar{\omega}_0(\rho) d\rho.$$

For any function  $\bar{\omega}_0 \in C([0, 1])$  we have

$$(2.2a) \quad \operatorname{div} \bar{u} = 0 \quad \text{in } \bar{\Omega},$$

$$(2.2b) \quad \bar{u} \cdot n = 0 \quad \text{on } \partial\Omega,$$

$$(2.2c) \quad \operatorname{rot} \bar{u} = \bar{\omega}_0 \quad \text{in } \bar{\Omega},$$

$$(2.2d) \quad (\bar{u}, \nabla) \bar{u} = - \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \frac{\bar{\varphi}^2}{r^3} = \nabla \bar{F} \quad \text{in } \bar{\Omega},$$

where  $\bar{\varphi}(r) = \int_0^r \rho \bar{\omega}_0(\rho) d\rho$  and  $\bar{F}(r) = - \int_0^r \bar{\varphi}^2(s)/s^3 ds$  which is well defined in  $[0, 1]$ , since

$$(2.3) \quad |\bar{\varphi}(s)|^2 \leq \int_0^s \rho^2 d\rho \cdot \int_0^s \bar{\omega}^2(\rho) d\rho \leq \frac{1}{3} s^3 \|\bar{\omega}\|_{L^2(0,1)}^2.$$

Thus,  $(\bar{u}, \bar{p})$  is the solution of (EE) for  $\bar{f} = 0$ , if  $\bar{u}$  is in (2.1) and  $\nabla \bar{p} = -\nabla \bar{F}$ .

We construct a non-stationary solution of (NS) in the form:

$$(2.4) \quad u^\nu(x, t) = \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix} \frac{1}{r} \int_0^r \rho \omega^\nu(\rho, t) d\rho,$$

where  $\omega^\nu(r, t)$  is unknown. We note that  $u^\nu(x, t)$  in (2.4) satisfies the same identities as (2.2a)–(2.2d).

To construct  $u^\nu(t)$ , we reduce (NS) to an equation of

$$(2.5) \quad \varphi^\nu(r, t) = \int_0^r \rho \omega^\nu(\rho, t) d\rho$$

instead of  $\omega^\nu = \operatorname{rot} u^\nu$ . By (2.2d) we have

$$\begin{aligned} & u_t^\nu - \nu \Delta u^\nu + (u^\nu, \nabla) u^\nu + \nabla p^\nu \\ &= \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix} \frac{1}{r} (\varphi_t^\nu - \nu \varphi_{rr}^\nu + \frac{\nu}{r} \varphi_r^\nu) + \nabla (F^\nu + p^\nu) = 0, \end{aligned}$$

where  $F^\nu = - \int_0^r (\varphi^\nu)^2(s, t)/s^3 ds$  which is well defined because of (2.3), if  $\omega^\nu \in L^\infty(0, T; L^2(\Omega))$ . Since a vector field  $\begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix} \Phi(r)$  is solenoidal for a radially symmetric function  $\Phi(r)$ , that is,

$$\operatorname{div} \left\{ \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix} \Phi(r) \right\} = 0 \quad \text{in } \Omega \quad \text{and} \quad \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix} \Phi(r) \cdot n = 0 \quad \text{on } \partial\Omega,$$

then the equation of  $\varphi^\nu$  is

$$(E) \quad \begin{aligned} \varphi_t^\nu - \nu \varphi_{rr}^\nu + \frac{\nu}{r} \varphi_r^\nu &= 0 \quad \text{for } (r, t) \in Q_T \equiv (0, 1) \times (0, T), \\ \varphi_r^\nu|_{r=0} &= 0, \quad \varphi_r^\nu|_{r=1} = 0 \quad \text{for } t \in (0, T), \\ \varphi^\nu|_{t=0} &= \varphi_0^\nu \equiv \int_0^r \rho \omega_0^\nu(\rho) d\rho \quad \text{for } r \in (0, 1), \end{aligned}$$

here  $\omega_0^\nu = \text{rot } u_0^\nu$  is given data,  $T$  is any but fixed positive number and a subscript of  $\varphi^\nu$  denotes partial differential with respect to its variable.

Thus let  $\varphi^\nu(t)$  be a solution of (E). Then  $(u^\nu, p^\nu)$  is a solution of (NS) for  $f^\nu = 0$  and  $u_0^\nu = {}^t(-\sin \theta, \cos \theta) \varphi_0^\nu / r$ , if  $u^\nu(t)$  is defined by (2.4) and  $p^\nu(t)$  is a solution of

$$\begin{aligned} \Delta p^\nu &= -\Delta F^\nu \quad \text{in } \Omega, \\ \nabla p^\nu \cdot n &= -\nabla F^\nu \cdot n \quad \text{on } \partial\Omega. \end{aligned}$$

For existence of a solution to (E) we have

**THEOREM 2 (EXISTENCE OF THE FLOW).** Assume

$$\omega_0^\nu(r) = \frac{1}{r} \partial_r(\varphi_0^\nu)(r), \quad \text{that is, } \varphi_0^\nu(r) = \int_0^r \rho \omega_0^\nu(\rho) d\rho$$

for  $\varphi_0^\nu \in C^{2+\alpha}([0, 1])$  with  $\varphi_0^\nu(0) = \partial_r(\varphi_0^\nu)(0) = 0$  and  $0 < \alpha < 1$ . Then there exists an unique solution  $\varphi^\nu \in C^{2,1}(Q)$  of (E), which satisfies

$$\begin{aligned} \varphi^\nu(0, t) &= 0 \quad \text{for } 0 \leq t < \infty, \\ |\varphi^\nu(r, t)| &\leq \frac{\sqrt{3}}{3} \|\omega_0^\nu\|_{L^2(0,1)} \quad \text{in } Q, \end{aligned}$$

where  $Q = \{(r, t); 0 \leq r \leq 1, 0 \leq t < \infty \text{ and } (r, t) \neq (1, 0)\}$  and  $\partial_r$  denotes the differential operator  $\frac{d}{dr}$ .

Here  $C^{2,1}(Q)$  (resp.  $C^{2+\alpha}([0, 1])$ ) is the Banach space whose elements have second derivatives in  $r$  and first derivatives in  $t$  (resp. second derivatives in  $r$ ). Furthermore second derivatives of the elements in  $C^{2+\alpha}([0, 1])$  are Hölder continuous with exponent  $\alpha$  in  $r \in [0, 1]$ .

**Remark.** In Theorem 2 we don't require the compatibility condition  $\varphi_0^\nu(1) = 0$ . Thus for the existence of a solution  $u^\nu(t)$  in (2.4) to (NS) we don't need to assume  $u_0^\nu|_{\partial\Omega} = 0$ . Hence our solution  $u^\nu(t)$  has the initial layer.

Finally our example is

**THEOREM 3 (CONVERGENCE OF THE FLOW).** Assume the same in Theorem 2 and  $\bar{\omega}_0 \in C([0,1])$  in (2.1). We put  $u_0^\nu = (-\sin\theta, \cos\theta)\varphi_0^\nu/r$  and let  $\bar{u}$  and  $u^\nu(t)$  be in (2.1) and (2.4) respectively. Finally we assume that  $u_0^\nu \rightarrow \bar{u}_0$  in  $L^2(\Omega)$  as  $\nu \rightarrow 0$  and  $\|\omega_0^\nu\|_{L^2(0,1)} \leq C$  independent of the viscosity  $\nu$ . Then we obtain for any but fixed  $T > 0$

$$\|u^\nu(t) - \bar{u}\|_{L^2(\Omega)} \rightarrow 0 \quad \text{as } \nu \rightarrow 0 \text{ uniformly in } t \in [0, T].$$

**Remark.** (1) Since we don't require  $\varphi_0^\nu(1) = 0$ , we can take  $\bar{u}_0$  as the initial data of (NS). (2) If we assume that the compatibility condition  $\varphi_0^\nu(1) = 0$  in Theorem 2, then by arguments likely to the below we can obtain

$$u^\nu \rightarrow \bar{u} \quad \text{in } C(K) \quad \text{as } \nu \rightarrow 0$$

for any compact subset  $K \subset \bar{Q}$ , even if  $|\omega_0^\nu(r)| \leq \nu^{-\epsilon}$  for  $1 - \nu^{2\epsilon} \leq r \leq 1$  and  $\epsilon < 1$  fixed. The proof is omitted in this report.

The remaining part in this section is to prove Theorem 3.

We denote by  $\psi(r, t)$ , the solution of

$$\begin{aligned} \psi_t - \psi_{rr} + \frac{1}{r}\psi_r &= 0 \quad \text{for } (r, t) \in Q_T, \\ \psi_r|_{r=0} &= 0, \quad \psi|_{r=1} = 0 \quad \text{for } t \in (0, T), \\ \psi|_{t=0} &= \varphi_0^\nu \equiv \int_0^r \rho \omega_0^\nu(\rho) d\rho \quad \text{for } r \in (0, 1). \end{aligned} \tag{E'}$$

Then the uniqueness of the solution to (E) implies

**LEMMA 1.** Let  $\varphi^\nu(t)$  be the solution of (E) in Theorem 2. Then we obtain

$$\varphi^\nu(r, t) = \psi(r, \nu t) \quad \text{in } Q$$

for a fixed  $\nu$ .

The following lemma plays the essential role in the proof of Theorem 3.

LEMMA 2. Let  $\varphi^\nu(t)$  be the solution in Theorem 2. Then

$$\left| \int_0^t \varphi_r^\nu(1, \tau) d\tau \right| \leq C(\|\omega_0^\nu\|_{L^2(0,1)} + 1) \exp C(\|\omega_0^\nu\|_{L^2(0,1)} T + 1)$$

for any  $t \in [0, T]$ , where  $C$  denotes several different positive constants independently of  $\nu$  and  $T$  here and after.

Proof. Let  $\psi(t)$  be in Lemma 1 and  $\nu$  be fixed. In (E') we replace  $t$  by  $\nu t$ . Then it follows that

$$\psi_{rr}(r, \nu t) - \frac{1}{r} \psi_r(r, \nu t) - \psi_t(r, \nu t) = 0 \quad \text{in } Q.$$

To integrate this equation in  $t$  on  $(\varepsilon, t)$  for any but fixed  $\varepsilon > 0$ , then  $f(r, t) = \int_\varepsilon^t \psi(r, \nu \tau) d\tau$  satisfies

$$\begin{aligned} f_{rr} - \frac{1}{r} f_r - f_t &= a_\varepsilon \quad \text{in } Q_\varepsilon^t = (0, 1) \times (\varepsilon, T), \\ f_r|_{r=0} &= 0, f|_{r=1} = 0 \quad \text{for } t \in (\varepsilon, T), \\ f|_{t=\varepsilon} &= 0 \quad \text{for } r \in (0, 1), \end{aligned}$$

where  $a_\varepsilon(r) = \psi(r, \nu \varepsilon)$ .

For  $\chi \in C^\infty(\mathbb{R})$  which satisfies  $0 \leq \chi(r) \leq 1$ ,  $\chi = 1$  in  $[2/3, \infty)$  and  $\chi = 0$  in  $(-\infty, 1/3]$ , we put  $z(t) = \chi^2 \exp f(t)$  and  $Pz = z_{rr} - z_t$ . Then we have

$$\begin{aligned} Pz &= (\chi^2)'' e^f + 4\chi\chi' f_r e^f + \chi^2 f_r^2 e^f + \chi^2 f_{rr} e^f - \chi^2 f_t e^f \\ &= e^f \{ \chi^2 (f_{rr} - f_t) + (\chi^2)'' + 4\chi\chi' f_r + \chi^2 f_r^2 \} \\ &= e^f \left\{ \frac{\chi^2}{r} f_r + \chi^2 a_\varepsilon + (\chi^2)'' + 4\chi\chi' f_r + \chi^2 f_r^2 \right\}. \end{aligned}$$

Since absolute values of  $\chi^2/r$ ,  $\chi'$  and  $(\chi^2)''$  are estimated by  $C$  for  $r \in [0, 1]$ , we obtain

$$Pz \geq C e^f \left\{ -\frac{1}{\mu} - \mu \chi^2 f_r^2 - \chi^2 |a_\varepsilon| - 1 - \frac{1}{\mu} - \mu \chi^2 f_r^2 + \chi^2 f_r^2 \right\}$$

for any  $\mu > 0$ .

Using the estimate  $|\psi(r, \nu t)| \leq (1/\sqrt{3})\|\omega_0\|_{L^2(0,1)}$  for any  $(r, t) \in Q$  in Theorem 2 and taking  $\mu = 1/2$ , then

$$\begin{aligned} Pz &\geq -C(\|\omega_0^\nu\|_{L^2(0,1)} + 1) \exp(C\|\omega_0^\nu\|_{L^2(0,1)}T) \\ &\equiv -M_1 e^{M_2}. \end{aligned}$$

Putting  $y = z + 2M_1 \exp(M_2 + r)$ , then  $Py > 0$  holds. Hence the maximum principle implies  $y(t)$  does not take its maximum in  $Q_T^\varepsilon \equiv [0, 1] \times [\varepsilon, T]$  at  $(r, t) \in (0, 1) \times (\varepsilon, T]$ . On the other hand, at parabolic boundary of  $Q_T^\varepsilon$ ,  $y(t)$  holds as follows:

$$\begin{aligned} y|_{r=0} &= z|_{r=0} + 2M_1 e^{M_2} = 2M_1 e^{M_2}, \\ y|_{t=\varepsilon} &= z|_{t=\varepsilon} + 2M_1 e^{M_2+r} = 2M_1 e^{M_2+r}, \\ y|_{r=1} &= z|_{r=1} + 2M_1 e^{M_2+1} = 2M_1 e^{M_2+1}, \end{aligned}$$

Hence at the all points  $(1, t)$  with  $\varepsilon \leq t \leq T$ ,  $y(r, t)$  attains its maximum in  $Q_T^\varepsilon$ . Then we conclude that

$$\frac{\partial y}{\partial r}|_{r=1} = \int_\varepsilon^t \psi_r(r, \nu \tau) d\tau + 2M_1 e^{M_2+1} \geq 0.$$

Putting  $\varepsilon \rightarrow 0$ , then

$$\int_0^t \psi_r(r, \nu \tau) d\tau \geq -2M_1 e^{M_2+1}.$$

The estimate from above of  $f(r, t)$  with  $\varepsilon = 0$  can be established in a similar way by making the substitution  $\hat{z} = \chi^2 \exp(-f)$  and considering  $\hat{y} = \hat{z} - 2M_1 \exp(M_2 + r)$ .

Hence by the identity in Lemma 1 the proof of our estimate is completed.

Finally we note that in this proof we use the method of the proof to Lemma 3 of Section 3 in [6].  $\square$

Now we show Theorem 3. Since

$$\begin{aligned} \text{rot } u^\nu|_{\partial\Omega} &= \omega^\nu(r, t)|_{r=1} = \frac{\partial}{\partial r} \int_0^r \rho \omega^\nu(\rho, t) d\rho|_{r=1} \\ &= \varphi_r^\nu(1, t), \end{aligned}$$



we have

$$\bar{u} \cdot n \times \operatorname{rot} u^\nu(t)|_{\partial\Omega} = -\varphi_\tau^\nu(1, t) \int_0^1 \rho \bar{\omega}_0(\rho) d\rho.$$

Thus we obtain an identity

$$\nu \int_0^t \int_{\partial\Omega} \bar{u} \cdot n \times \operatorname{rot} u^\nu(\tau) dS d\tau = -2\pi\nu \int_0^1 \rho \bar{\omega}_0(\rho) d\rho \cdot \int_0^t \varphi_\tau^\nu(1, \tau) d\tau.$$

Hence it is easy to show that (c) in Theorem 1 holds because of Lemma 2. This proves Theorem 3 by Theorem 1.

For the proofs of Theorem 1 and Theorem 2, see [5].

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